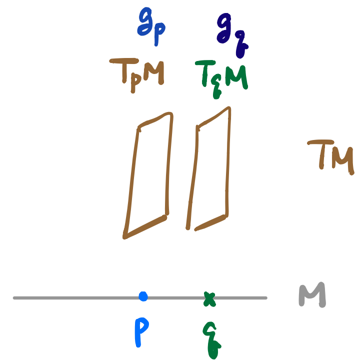
MATH 5061 Lecture on 3/18/2020



[Announcement: HW3 due today. HW4 posted.]

Recall: (M^m, g) Riemannian manifold

is • a C^∞ mfd M^m

• $\exists$ fiber (pos. def.) metric $\overset{\langle , \rangle}{\overset{\shortparallel}{g}}$ on $TM \longrightarrow M$.

Fundamental Thm. of Riem. Geometry

Given a Riem. mfd (M^m, g), $\exists!$ connection D on TM s.t.

(1) $Dg \equiv 0$ — metric compatible

(2) $T \equiv 0$ — torsion free

$\Big\}$ Riemannian / Levi-Civita connection

Proof: (Constructive proof)

GOAL: Derive an explicit formula of D using ONLY (1), (2).

For $X, Y, Z \in \mathfrak{X}(M)$,

$$X(\langle Y, Z\rangle) \overset{(1)}{=} \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$$

$$+)\quad Y(\langle Z, X\rangle) \overset{(1)}{=} \langle D_Y Z, X\rangle + \langle Z, D_Y X\rangle$$

$$-)\quad Z(\langle X, Y\rangle) \overset{(1)}{=} \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle$$

$$\text{R.H.S.} = \langle D_X Y + D_Y X, Z\rangle + \langle D_X Z - D_Z X, Y\rangle + \langle D_Y Z - D_Z Y, X\rangle$$

$$\overset{(2)}{=} \langle 2 D_X Y - [X, Y], Z\rangle + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle$$

Rearrange the terms,

$$\underbrace{2\langle D_X Y, Z\rangle}_{\text{defines } D_X Y} = \underbrace{\langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle - Z(\langle X,Y\rangle) + Y(\langle X,Z\rangle) + X(\langle Y,Z\rangle)}_{\text{only involves } \langle , \rangle = g \text{ and } [\cdot,\cdot]}$$

locally in coordinates:

$$\boxed{\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}\left(g_{\ell j,i} + g_{i\ell,j} - g_{ij,\ell}\right)}$$

Koszul's formula

$\square$

Remark: $T_{ij}^k \approx F(g, \partial g)$.

By general connection theory,

D Levi-Civita connection on TM $\rightsquigarrow$ Riemann curvature $\quad \text{Riem} = R$

$\forall X, Y \in \mathfrak{X}(M)$, $\quad R(X,Y) : TM \to TM$, i.e. $R \in \Gamma(\Lambda^2 T^*M \otimes \text{End}(TM))$

locally $\quad R = \{\Omega^i_j\}$, $\quad \Omega^i_j$: local 2-forms on M

Suppose: $e_1, \dots, e_m$ basis for TM

$\theta^1, \dots, \theta^m$ dual basis for T^*M

we can write $\quad \Omega^i_j = \frac{1}{2} \sum_{k,l} R^i_{jkl} \, \theta^k \wedge \theta^l \quad$ where $R^i_{jkl} = -R^i_{jlk}$

lower its index: $\quad \Omega_{ij} := \sum_p g_{ip} \Omega^p_j = \frac{1}{2} \sum_{k,l} R_{ijkl} \, \theta^k \wedge \theta^l$.

Symmetries of R_{ijkl}:

✓(1) $R_{ijkl} = -R_{ijlk} = -R_{jikl}$

✓(2) $R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad$ (1st Bianchi identity)

✓(3) $R_{ijkl} = R_{klij}$

Pf: (1) 1st "=" by def^{n}, 2nd "=" $\because$ $\Omega_{ji} = -\Omega_{ij}$ for metric compatible D.

(2) Locally in basis: $\underline{\underline{e}} = (e_1, \dots, e_m)$; $\underline{\underline{\theta}} = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^m \end{pmatrix}$.

$$D\underline{\underline{e}} = \underline{\underline{e}}\,\omega \quad \underset{\text{(HW)}}{\Rightarrow} \quad d\underline{\underline{\theta}} = -\omega \wedge \underline{\underline{\theta}}$$

So, $\quad 0 = d^2 \underline{\underline{\theta}} = -d\omega \wedge \underline{\underline{\theta}} + \omega \wedge d\underline{\underline{\theta}}$

$$= -d\omega \wedge \underline{\underline{\theta}} + \omega \wedge (-\omega \wedge \underline{\underline{\theta}})$$

$$= -\underbrace{(d\omega + \omega \wedge \omega)}_{=:\Omega} \wedge \underline{\underline{\theta}}$$

Thus, we obtained $\boxed{\Omega \wedge \underline{\underline{\theta}} = 0}$

More explicitly, $\frac{1}{2} \sum_{j,k,l} R^i_{jkl}\, \theta^k \wedge \theta^l \wedge \theta^j = 0$

Look at the components,

$$\Rightarrow \left(R^i_{jkl} + R^i_{klj} + R^i_{ljk} \right) - \left(R^i_{kjl} + R^i_{jlk} + R^i_{lkj} \right) = 0$$

lower index $\Rightarrow \left(R_{ijkl} + R_{iklj} + R_{iljk} \right) - \left(R_{ikjl} + R_{ijlk} + R_{ilkj} \right) = 0$

$$\Rightarrow \quad 2\left(R_{ijkl} + R_{iklj} + R_{iljk} \right) = 0$$

(1) + (2) $\Rightarrow$ (3) :

$$R_{ijkl} + R_{iklj} + R_{iljk} \overset{(2)}{=} 0$$

$$-) \quad R_{jikl} + R_{jkli} + R_{jlik} \overset{(2)}{=} 0$$

$$2 R_{ijkl} + \underbrace{R_{iklj} + R_{iljk} + R_{jkil} + R_{ljik}}_{\substack{\text{invariant under} \\ (i,j) \leftrightarrow (k,l)}} = 0$$

$\square$

Remarks: $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ $\quad R$: (0,4) – tensor

$$R(X,Y)\, e_j = \Omega^i_j(X,Y)\, e_i \Rightarrow \Omega_{ij}(X,Y) = \langle R(X,Y) e_j, e_i \rangle$$

i.e. $R(X,Y,Z,W) = -\langle R(X,Y)Z, W\rangle$

In local coord. $\boxed{R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^p_{lj}\Gamma^i_{kp} - \Gamma^p_{kj}\Gamma^i_{lp}}$ (Ex:)

Remark: $g = (g_{ij}) \rightsquigarrow D = \{\Gamma^k_{ij}\} = F(g, \partial g)$

$\rightsquigarrow R = \{R_{ijkl}\} = \tilde{F}(g, \partial g, \partial^2 g)$

One consequence of the symmetries of R_{ijkl}

At $p \in M$, $\mathcal{R}_p : \Lambda^2 T_pM \times \Lambda^2_p TM \to \mathbb{R}$

s.t. $\mathcal{R}_p(e_i \wedge e_j, e_k \wedge e_l) := R_{ijkl}$

Curvature operator acting on $\Lambda^2 TM$ (or $\Lambda^2 T^*M$)

(1) $\Rightarrow$ well-defined

(3) $\Rightarrow$ symmetric

Def$^{\underline{n}}$: $F:(M^m,g)\longrightarrow(N^m,h)$ is an isometry

if $F:M\to N$ is diffeomorphism & $F^*h=g$.

Remark: Riem. is a "geometric invariant" (indep. of coord.)

$$F:(M,g)\longrightarrow(N,h) \quad\Rightarrow\quad F^*(\mathrm{Riem}(h))=\mathrm{Riem}(g)$$

isometry

i.e. $F^*h=g$

E.g.) "Locally" isometric manifolds have the "same" Riem. curvature

Q: How much does Riem. curvature determine the metric?

"**A**": In general, it's not fully. But sometimes true.

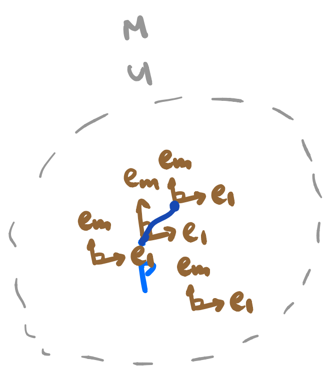
Thm: (M^m,g) $\mathrm{Riem}(g)\equiv 0 \iff (M^m,g)$ is locally isometric to $(\mathbb{R}^m,\delta)$

standard Euclidean metric

Proof: ($\Leftarrow$) trivial.

($\Rightarrow$) Suppose (M^m,g) with $\mathrm{Riem}(g)\equiv 0$, i.e. D is flat.

Fix $p\in M$, and $e_1,\dots,e_m$ O.N.B. of T_pM.



D flat $\Rightarrow$ $\exists$ parallel extension of $e_1,\dots,e_m$ (i.e. $De_i\equiv 0$ near p)

in a nbd. of p in M

Let $g_{ij}:=\langle e_i,e_j\rangle \underset{\text{parallel}}{\equiv} \langle e_i,e_j\rangle(p)=\delta_{ij}$

GOAL: Show $\exists$ local coord. $x^1,\dots,x^m$ of M s.t. $e_i=\frac{\partial}{\partial x^i}$.

Consider dual basis of 1-forms, $\theta^1,\dots,\theta^m$ of $e_1,\dots,e_m$

Recall: $d\underline{\underline{\theta}}=-\omega\wedge\underline{\underline{\theta}} \overset{D\text{ flat}}{=} 0$, i.e. $d\theta^i\equiv 0$

By "Poincaré lemma", $\theta^i=dx^i$ for some fcn x^i near p

So, $\langle\theta^i,\theta^j\rangle\equiv\delta_{ij} \Rightarrow F:(M,g)\longrightarrow(\mathbb{R}^m,\delta)$ is local isometry.

$q\longmapsto(x^1(q),\dots,x^m(q))$ $\square$

Remark: Globally may have topology, e.g. flat torus $T^m = \mathbb{R}^m/\Lambda$

but the thm. is true passing to the universal cover $\tilde{M}$

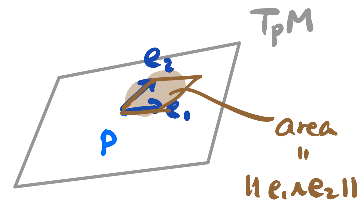
Note: Riem. curvature Riem of (M^m, g) is the higher dimensional analogue to the notion of Gauss curvature for surfaces (in $\mathbb{R}^3$).

- When $m=2$ (surface case),

$$\mathcal{R}_p : \Lambda^2 T_pM \times \Lambda^2 T_pM \longrightarrow \mathbb{R} \qquad \text{say } \Lambda^2 T_pM = \operatorname{span}\{e_1 \wedge e_2\}$$

Gauss curvature at p ⇝

$$K_p = \mathcal{R}_p\left(\frac{e_1\wedge e_2}{\|e_1\wedge e_2\|}, \frac{e_1\wedge e_2}{\|e_1\wedge e_2\|}\right) = \frac{R_{1212}}{\|e_1\wedge e_2\|^2}$$



In general, we have

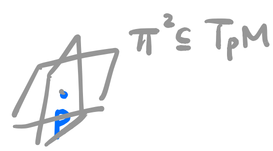
Def^n: Let $\pi^2 \subseteq T_pM$ be a 2-dim. subspace w/. basis $\{v, w\}$.

$$K_p(\pi) := \frac{\mathcal{R}(v\wedge w, v\wedge w)}{\|v\wedge w\|^2}$$

section curvature of π at p

Note: $K_p(\pi)$ is indep. of choice of basis $\{v,w\}$ for π.

$$K_p : \underbrace{Gr(2, T_pM)}_{\{\pi \subseteq T_pM \,:\, \pi \text{ 2-dim subspace}\}} \longrightarrow \mathbb{R}$$



locally: $e_1, \dots, e_m$ O.N.B. of T_pM $\qquad K_p(\operatorname{span}\{e_i, e_j\}) = R_{ijij}$.

Q: Does sectional curvature determine Riem completely?

A: Yes!

Thm: The sectional curvatures determines Riem. curvature tensor.

i.e. If $R^{(1)}, R^{(2)}$ are $(0,4)$-tensors satisfying the symmetries (1) - (3) of Riem.

then $R^{(1)} = R^{(2)}$.

Note: This is just an algebraic statement:

knowing R_{ijij} $\underset{\text{symm.}}{\overset{\text{w/.}}{\rightsquigarrow}}$ knowing R_{ijkl}

Proof: Let $R = R^{(1)} - R^{(2)}$, then R satisfies the symmetries (1)–(3).

sectional curvatures of $R^{(1)}, R^{(2)}$ agree. $\Rightarrow$ $R(X,Y,X,Y) = 0$ $\forall X,Y$ o.n. of T_pM.

To show $R(X,Y,Z,W) = 0$ $\forall X,Y,Z,W$

it suffices to show the case $(X,Y),(Z,W)$ are mutually O.N.

Claim 1: $R(X,Y,Z,Y) = 0$ (i.e. $Y = W$).

$$0 = R\left(\frac{X+Z}{\sqrt{2}}, Y, \frac{X+Z}{\sqrt{2}}, Y\right) = \frac{1}{2} R(X+Z, Y, X+Z, Y) = R(X,Y,Z,Y)$$

$\Rightarrow$ case $m=3$ true.

Claim 2: $R(X,Y,Z,W) = 0$ for $m \geq 4$.

$$0 = R\left(X, \frac{Y+W}{\sqrt{2}}, Z, \frac{Y+W}{\sqrt{2}}\right) = \frac{1}{2} R(X, Y+W, Z, Y+W)$$

by Claim 1

$$= \frac{1}{2}\left(R(X,Y,Z,W) + R(X,W,Z,Y)\right)$$

$$\Rightarrow R(X,Y,Z,W) = -R(X,W,Z,Y) = R(X,W,Y,Z)$$

$$= -R(X,Z,Y,W) = R(X,Z,W,Y) = 0$$

by 1st Bianchi identity.

$\square$

Cor: g has constant sectional curvature $\equiv K_0 \in \mathbb{R}$ everywhere.

$$\Rightarrow \quad R_{ijkl} = K_0 (g_{ik} g_{jl} - g_{il} g_{jk})$$

Next: Riem is a $(0,4)$-tensor which admits a natural algebraic decomposition into 3 parts:

$$\text{Riem} = \text{"}W\text{"} + \text{"}\overset{\circ}{\text{Ric}}\text{"} + \text{"}R\text{"}$$

Weyl curvature; trace-free Ricci curv; scalar curvature

By take traces of the (0,4)-tensor Riem = $R_{ijk\ell}$:

Def$^{\underline{n}}$: $\mathrm{Ric}(X,Y) := \sum_{i=1}^{m} R(X, e_i, Y, e_i)$ ← Ricci curvature

where $e_1, \dots, e_m$ O.N.B. for T_pM

$\mathrm{Scal} = R := \sum_{i=1}^{m} \mathrm{Ric}(e_i, e_i)$ ← Scalar curvature

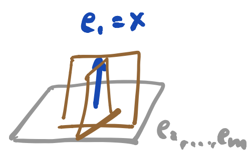
Observations: • Ric is a (0,2)-tensor, R is a function.

- locally: $\mathrm{Ric} = R_{j\ell} = R^{i}{}_{ji\ell} \quad (\overset{\text{O.N.B.}}{=} R_{iji\ell} = R_{ji\ell i})$

  $R = R^{j}{}_{j} = R^{ij}{}_{ij} \quad (\overset{\text{O.N.B.}}{=} R_{ijij})$

- $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X) \Rightarrow$ Ric is a symm. (0,2)-tensor (like g)

- $m=2$: $R = 2K$.



- $\mathrm{Ric}(X,X) := \sum_{i=1}^{m} R(X, e_i, X, e_i) = \sum_{j=2}^{m} \underbrace{R(e_1, e_j, e_1, e_j)}_{\text{sect. curv of span}\{e_1, e_j\}}$

  say $X = e_1, e_2, \dots, e_m$ O.N.B.

So, $\mathrm{Ric}(X,X) \approx$ average sectional curv. of $\Pi \ni X$.

Recall: For any symm. (0,2)-tensor h. we have a decomposition

$$h = \underbrace{\mathring{h}}_{\substack{\text{trace-}\\\text{free}\\\text{part}}} + \underbrace{\frac{\mathrm{Tr}(h)}{m} g}_{\substack{\text{trace}\\\text{part}}}$$

Think in terms of matrix:

$$A = \mathring{A} + \frac{\mathrm{Tr}(A)}{m} I$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -1/2 & 2 \\ 2 & 1/2 \end{pmatrix} + \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

$\mathrm{tr}A = 3 \qquad \mathrm{tr}\mathring{A} = 0 \qquad \mathrm{tr} = 3$

Def$^{\underline{n}}$: (Kulkarni-Nomizu Product $\circ$)

$$\underbrace{(h \circ P)}_{\text{symm. (0,2)-tensors}}{}_{ijkl} := \underbrace{h_{ik}P_{jl} + h_{jl}P_{ik} - h_{jk}P_{il} - h_{il}P_{jk}}_{\text{same symmetries as } R_{ijkl}}$$

Then,

$$\mathrm{Riem} = W + \frac{1}{m-2}\,\mathring{\mathrm{Ric}} \circ g + \frac{R}{2m(m-1)}\, g \circ g$$

scalar curvature

Remarks:
- W_{ijkl} is "trace-free" (i.e. $W_{ijil} = 0$)
- $\mathring{\mathrm{Ric}} := \mathrm{Ric} - \frac{R}{m} g$ is the trace-free Ricci tensor

Prop: (Weyl curvature tensor)

(i) $m=2$; then $W \equiv 0$, $\mathring{\mathrm{Ric}} \equiv 0$, and $R = 2K$.

(ii) $m=3$: $W \equiv 0$ (ie. Riem is determined by Ric)

(iii) W is "conformally invariant", i.e

$$\tilde{g} = e^{2u} g \implies \tilde{W} = e^{2u} W$$

for some $u \in C^\infty(M)$

Proof: HW problems.

Remark: When $m \geq 4$, $W_g \equiv 0 \iff g$ is "locally conformally flat"

(ie $\exists$ coord system s.t $g_{ij} = e^{2u}\delta_{ij}$)